Stochastic Control Representations for Penalized Backward Stochastic Differential Equations

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Abstract

We show that both reflected backward stochastic differential equation (*reflected BSDE* for short) and its associated penalized backward stochastic differential equation (*penalized BSDE* for short) admit both optimal stopping representation and optimal control representation. We also show that both multidimensional reflected BSDE and its associated multidimensional penalized BSDE admit optimal switching representation. The corresponding optimal stopping (switching) problems for penalized BSDE have the feature that only stopping at exogenous Poisson arrival times is allowed.

Keywords: Reflected BSDE, penalized BSDE, optimal stopping time, optimal control, optimal switching.

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1 Introduction

El Karoui et al [5] introduced penalized backward stochastic differential equation (*penalized BSDE* for short) to solve reflected backward stochastic differential equation (*reflected BSDE* for short), and they showed that the solution of a reflected BSDE corresponds to the value process of a nonlinear optimal stopping time problem. In this paper, our main result is to show that the solution of the associated penalized BSDE also corresponds to the value process of some nonlinear optimal stopping time problem, and the parameter λ appearing in the penalized equation is nothing but the intensity of some exogenous Poisson process.

Let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ satisfying the *usual conditions*, i.e. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and complete. In El Karoui et al [5], the authors introduced the following reflected BSDE:

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds + \int_t^T dK_s - \int_t^T Z_s dW_s \tag{1.1}$$

under the constraints:

$$\begin{aligned} &\text{(Dominating Condition)}: \quad Y_t \geq S_t \text{ for } t \in [0, T], \\ &\text{(Skorohod Condition)}: \quad \int_0^T (Y_t - S_t)dK_t = 0 \text{ for } K \text{ continuous and increasing,} \end{aligned}$$

where the terminal data ξ, the driver $f_s(y, z)$, and the obstacle $(S_t)_{0\leq t\leq T}$ are the given data for the equation. A solution to reflected BSDE (1.1) is a triplet (Y, Z, K), where K is a kind of "local time" process. The equation (1.1) corresponds to a backward Skorohod problem, which in turn gives the "local time" process K a Skorohod representation. See Qian and Xu [18] in this direction.

On the other hand, as shown in [5], (1.1) also has an interesting interpretation in the sense that its solution is the value process of a nonlinear optimal stopping time problem: For any fixed time $t \in [0, T]$, the value process of the following optimal stopping time problem:

$$y_t = \text{esssup}_{\tau\in\mathcal{R}(t)}\mathbf{E}\left[\int_t^{\tau\wedge T} f_s(Y_s, Z_s)ds + S_\tau \mathbf{1}_{\{\tau<T\}} + \xi\mathbf{1}_{\{\tau\geq T\}}|\mathcal{F}_t\right], \tag{1.2}$$

where the control set $\mathcal{R}(t)$ is defined by

$$\mathcal{R}(t) = \{\mathcal{F}_s\text{-stopping times } \tau \geq t\},$$

is given by the solution to reflected BSDE (1.1): $y_t = Y_t$ a.s. for $t \in [0, T]$. The optimal stopping time is given by $\tau^* = \inf\{s \geq t : Y_s = S_s\} \wedge T$. The nonlinear optimal stopping problem (1.2)

is closely related to pricing and hedging American options especially in constrained markets as shown in El Karoui et al [6].

One way to solve reflected BSDE (1.1) is to iterate the solution of the corresponding backward Skorohod problem by Picard iteration. The other way, which seems more commonly used in the literature, is to approximate the “local time” process K by

$$K_t^\lambda = \int_0^t \lambda \max\{0, S_s - Y_s^\lambda\} ds \text{ for } t \in [0, T],$$

where (Y^λ, Z^λ) is the solution of the following penalized BSDE:

$$Y_t^\lambda = \xi + \int_t^T f_s(Y_s^\lambda, Z_s^\lambda) ds + \int_t^T \lambda \max\{0, S_s - Y_s^\lambda\} ds - \int_t^T Z_s^\lambda dW_s. \tag{1.3}$$

Under Assumption 1.1 introduced below, El Karoui et al [5] proved that Y^λ is increasing in λ, and

$$\lim_{\lambda \uparrow \infty} \mathbf{E}\left[\sup_{t\in[0,T]} |Y_t^\lambda - Y_t|^2 + \int_0^T |Z_t^\lambda - Z_t|^2 dt + \sup_{t\in[0,T]} |K_t^\lambda - K_t|^2\right] = 0. \tag{1.4}$$

Our aim is to give stochastic control representations for penalized BSDE (1.3). Our main result is to prove that penalized BSDE (1.3) also admits an optimal stopping representation. We impose the following standard assumption on the data set (ξ, f, S) as in [5]:

Assumption 1.1

- *The terminal data ξ is $\mathbb{L}^2$-square integrable:* $\mathbf{E}[|\xi|^2] < \infty$;
- *The driver $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is uniformly Lipschitz continuous:*

$$|f_t(y, z) - f_t(\bar{y}, \bar{z})| \le C(|y - \bar{y}| + |z - \bar{z}|) \text{ a.s. for some } C > 0,$$

and $f_t(0, 0)$ is $\mathbb{H}^2$-square integrable: $\mathbf{E}\left[\int_0^T |f_t(0, 0)|^2 dt\right] < \infty$;

- *The obstacle process S is continuous and uniformly square integrable:* $\mathbf{E}\left[\sup_{t\in[0,T]} |S_t|^2\right] < \infty$.

The above conditions can be relaxed. See, for example, Peng and Xu [17] and Lepeltier and Xu [13] extending to RCLL obstacles, and Bayraktar and Song [1] and Lionnet [16] among others extending to the driver $f_s(y, z)$ with quadratic growth in z. However, we only stick with the above standard assumption in this paper. Under the above standard assumption, we have the following representation which is the main result of this paper.

Theorem 1.2 *Suppose that Assumption 1.1 holds. Denote $(Y^{\lambda}, Z^{\lambda})$ as the unique solution to penalized BSDE (1.3). For any fixed time $t \in [0, T]$, define the control set $\mathcal{R}(t, \lambda)$ as*

$$\mathcal{R}(t,\lambda) = \{\mathcal{G}_s\text{-stopping time } \tau \geq t : \tau = T_n \text{ for some } n \geq 1\},$$

where $\{T_n\}_{n\geq 0}$ are the arrival times of a Poisson process $(N_s)_{s\geq t}$ with intensity λ and minimal augmented filtration $\{\mathcal{H}_s\}_{s\geq t}$, and $\mathcal{G}_s = \mathcal{F}_s \vee \mathcal{H}_s$. Then the value process of the following optimal stopping time problem:

$$y_t^{\lambda} = esssup_{\tau\in\mathcal{R}(t,\lambda)}\mathbf{E}\left[\int_t^{\tau\wedge T} f_s(Y_s^{\lambda}, Z_s^{\lambda})ds + S_\tau \mathbf{1}_{\{\tau<T\}} + \xi\mathbf{1}_{\{\tau\geq T\}}|\mathcal{F}_t\right] \tag{1.5}$$

is given by the solution to penalized BSDE (1.3): $y_t^{\lambda} = Y_t^{\lambda}$ a.s. for $t \in [0, T]$. The optimal stopping time is given by $\tau^* = \inf\{T_n \geq t : Y_{T_n}^{\lambda} \leq S_{T_n}\} \wedge T$.

Note that the value process y^{λ} is increasing in λ, because the larger λ is, the more stopping opportunities we have. By the convergence (1.4) and Theorem 1.2, the two value processes defined in (1.2) and (1.5) are related by

$$\lim_{\lambda\uparrow\infty} \mathbf{E}\left[\sup_{t\in[0,T]} |y_t^{\lambda} - y_t|^2\right] = 0. \tag{1.6}$$

The proof of Theorem 1.2 is provided in Section 2. Based on the above representation result and the idea of randomized stopping by Krylov [12], we also give an optimal control representation for penalized BSDE, which is formulated and proved in Section 3. In Section 4, we extend our representation results to multidimensional case, and give an optimal switching representation for multidimensional penalized BSDE associated with multidimensional reflected (oblique) BSDE. Finally, Section 5 concludes.

2 Proof of Theorem 1.2

The optimal stopping time problem (1.5) has a constraint on its control set, i.e. the optimal stopping time must be chosen from the arrival times $\{T_n\}_{n\geq 1}$ of a Poisson process N. Given the arrival time T_n, by defining pre-T_n σ-field:

$$\mathcal{G}_{T_n}(t,\lambda) = \left\{A \in \bigvee_{s\geq t} \mathcal{G}_s : A \cap \{T_n \leq s\} \in \mathcal{G}_s \text{ for } s \geq t.\right\},$$

it is easy to show that the problem (1.5) is equivalent to the following discrete optimal stopping time problem where the control constraint does not appear:

$$y_t^\lambda = \mathrm{esssup}_{N\in\mathcal{N}(t,\lambda)}\mathbf{E}\left[\int_t^{T_N\wedge T} f_s(Y_s^\lambda, Z_s^\lambda)ds + S_{T_N}\mathbf{1}_{\{T_N<T\}} + \xi\mathbf{1}_{\{T_N\geq T\}}|\mathcal{F}_t\right], \tag{2.1}$$

where

$$\mathcal{N}(t,\lambda) = \left\{\mathcal{G}_{T_N}(t,\lambda)\text{-stopping time } N : N = n \text{ for some integer } n\geq 1\right\}.$$

Note that (2.1) is a discrete optimal stopping problem, as we are only allowed to stop at a sequence of deterministic times, the integers $(n)_{n\geq 1}$. The optimal stopping time is some integer N^* such that $\tau^* = T_{N^*}\wedge T$. If $T_{N^*}\geq T$, we define the optimal stopping time as terminal time T by convention.

2.1 Representation for Linear Case

In this section, we consider the case where the driver $f_s(y,z)$ is independent of (y,z), and simply write it as f_s in such a situation. Note that the corresponding reflected BSDE (1.1) becomes linear, and its optimal stopping time problem (1.2) also becomes linear.

The following lemma is the key to prove Theorem 1.2. Basically, it says that we only need to consider the optimal stopping time problem (2.1) up to the first Poisson arrival time T_1.

Lemma 2.1 *Suppose that Assumption 1.1 holds, and that $f_s(y,z) = f_s$. Then the value process of the optimal stopping time problem (2.1) satisfies the following dynamic programming equation:*

$$y_t^\lambda = \mathbf{E}\left[\int_t^{T_1\wedge T} f_s ds + \max\left\{S_{T_1}, y_{T_1}^\lambda\right\}\mathbf{1}_{\{t<T_1\leq T\}} + \xi\mathbf{1}_{\{T_1\geq T\}}|\mathcal{F}_t\right]. \tag{2.2}$$

Proof. We first introduce an auxiliary optimal stopping time problem:

$$\hat{y}_t^\lambda = \mathrm{esssup}_{N\in\hat{\mathcal{N}}(t,\lambda)}\mathbf{E}\left[\int_t^{T_N\wedge T} f_s ds + S_{T_N}\mathbf{1}_{\{T_N<T\}} + \xi\mathbf{1}_{\{T_N\geq T\}}|\mathcal{F}_t\right], \tag{2.3}$$

where

$$\hat{\mathcal{N}}(t,\lambda) = \left\{\mathcal{G}_{T_N}(t,\lambda)\text{-stopping time } N : N = n \text{ for some integer } n\geq 0\right\}.$$

The difference between (2.3) and (2.1) is that the former is allowed to stop at the starting time t, while the later not. Due to the additional stopping opportunity of problem (2.3), we have the following relationship between the two value processes:

$$\begin{aligned}\hat{y}_t^\lambda &= \max\left\{\mathrm{esssup}_{N\in\mathcal{N}(t,\lambda)}\mathbf{E}\left[\int_t^{T_N\wedge T} f_s ds + S_{T_N}\mathbf{1}_{\{T_N<T\}} + \xi\mathbf{1}_{\{T_N\geq T\}}|\mathcal{F}_t\right], S_t\right\}\\ &= \max\left\{y_t^\lambda, S_t\right\} \text{ for } t\in[0,T]. \end{aligned}\tag{2.4}$$

Moreover, if N^* is the optimal stopping time for problem (2.1) starting from time t, by time consistency and (2.4), N^* is also the optimal stopping time for problem (2.3) starting from the first Poisson arrival time T_1:

$$\hat{y}^{\lambda}_{T_1} = \mathbf{E}\left[\int_{T_1}^{T_{N^*}\wedge T} f_s ds + S_{T_{N^*}}\mathbf{1}_{\{T_{N^*}<T\}} + \xi\mathbf{1}_{\{T_{N^*}\geq T\}}|\mathcal{F}_{T_1}\right].$$

If N^* is the optimal stopping time for problem (2.1), by conditional on the first Poisson arrival time T_1, we have

$$\begin{aligned}
y^{\lambda}_t &= \mathbf{E}\left[\mathbf{E}\left[\int_t^{T_{N^*}\wedge T} f_s ds + S_{T_{N^*}}\mathbf{1}_{\{T_{N^*}<T\}} + \xi\mathbf{1}_{\{T_{N^*}\geq T\}}|\mathcal{F}_{T_1}\right]|\mathcal{F}_t\right] \\
&= \mathbf{E}\Big[\mathbf{E}\Big[\Big(\mathbf{1}_{\{t<T_1<T, T_1\leq T_{N^*}<T\}} + \mathbf{1}_{\{t<T_1<T, T_{N^*}\geq T\}} + \mathbf{1}_{\{T_1\geq T\}}\Big) \\
&\qquad \times \left(\int_t^{T_{N^*}\wedge T} f_s ds + S_{T_{N^*}}\mathbf{1}_{\{T_{N^*}<T\}} + \xi\mathbf{1}_{\{T_{N^*}\geq T\}}\right)|\mathcal{F}_{T_1}\Big]|\mathcal{F}_t\Big] \\
&= \mathbf{E}\Big[\mathbf{1}_{\{t<T_1<T\}}\mathbf{E}\left[\mathbf{1}_{\{T_1\leq T_{N^*}<T\}}\left(\int_t^{T_1} f_s ds + \int_{T_1}^{T_{N^*}} f_s ds + S_{T_{N^*}}\right)|\mathcal{F}_{T_1}\right] \\
&\qquad + \mathbf{1}_{\{t<T_1<T\}}\mathbf{E}\left[\mathbf{1}_{\{T_{N^*}\geq T\}}\left(\int_t^{T_1} f_s ds + \int_{T_1}^{T} f_s ds + \xi\right)|\mathcal{F}_{T_1}\right] \\
&\qquad + \mathbf{1}_{\{T_1\geq T\}}\mathbf{E}\left[\int_t^T f_s ds + \xi|\mathcal{F}_{T_1}\right]|\mathcal{F}_t\Big] \\
&= \mathbf{E}\left[\mathbf{1}_{\{t<T_1<T\}}I_1 + \mathbf{1}_{\{t<T_1<T\}}I_2 + \mathbf{1}_{\{T_1\geq T\}}I_3|\mathcal{F}_t\right],
\end{aligned}$$

where we used the fact that on $\{T_1 \geq T\}$, we must have $\{T_{N^*} \geq T\}$. Moreover,

$$I_1 = \mathbf{P}(T_1 \leq T_N^* < T|\mathcal{F}_{T_1})\int_t^{T_1} f_s ds + \mathbf{E}\left[\mathbf{1}_{\{T_1\leq T_{N^*}<T\}}\left(\int_{T_1}^{T_{N^*}} f_s ds + S_{T_{N^*}}\right)|\mathcal{F}_{T_1}\right],$$

and

$$I_2 = \mathbf{P}(T_N^* \geq T|\mathcal{F}_{T_1})\int_t^{T_1} f_s ds + \mathbf{E}\left[\mathbf{1}_{\{T_{N^*}\geq T\}}\left(\int_{T_1}^{T} f_s ds + \xi\right)|\mathcal{F}_{T_1}\right].$$

By plugging I_1 and I_2 into the expression for y^{λ}_t, we obtain

$$\begin{aligned}
y^{\lambda}_t = \mathbf{E}\Big[&\mathbf{1}_{\{t<T_1<T\}}\int_t^{T_1} f_s ds \\
&+ \mathbf{1}_{\{t<T_1<T\}}\mathbf{E}\left[\int_{T_1}^{T_{N^*}\wedge T} f_s ds + S_{T_{N^*}}\mathbf{1}_{\{T_1\leq T_{N^*}<T\}} + \xi\mathbf{1}_{\{T_{N^*}\geq T\}}|\mathcal{F}_{T_1}\right] \\
&+ \mathbf{1}_{\{T_1\geq T\}}\left(\int_t^T f_s ds + \xi\right)|\mathcal{F}_t\Big].
\end{aligned}$$

Note that the second term on the RHS of the above equality is

$$\text{esssup}_{N\in\hat{\mathcal{N}}(T_1,\lambda)}\mathbf{E}\left[\int_{T_1}^{T_N\wedge T} f_s ds + S_{T_N}\mathbf{1}_{\{T_N<T\}} + \xi\mathbf{1}_{\{T_N\geq T\}}|\mathcal{F}_{T_1}\right],$$

which by definition is $\hat{y}^{\lambda}_{T_1}$. The result then follows from the relationship (2.4). ■

Next, we make use of the density function $\lambda e^{-\lambda(x-t)}dx$ of the first Poisson arrival time T_1 to simplify dynamic programming equation (2.2).

$$\begin{aligned}\mathbf{E}\left[\int_t^{T_1\wedge T} f_s ds|\mathcal{F}_t\right] &= \mathbf{E}\left[e^{-\lambda(T-t)}\int_t^T f_s ds + \int_t^T \lambda e^{-\lambda(s-t)}(\int_t^s f_u du)ds|\mathcal{F}_t\right]\\ &= \mathbf{E}\left[\int_t^T e^{-\lambda(s-t)} f_s ds|\mathcal{F}_t\right],\end{aligned}$$

where we used integration by parts in the second equality. Moreover,

$$\begin{aligned}&\mathbf{E}\left[\max\left\{S_{T_1}, y^{\lambda}_{T_1}\right\}\mathbf{1}_{\{t<T_1\le T\}} + \xi\mathbf{1}_{\{T_1\ge T\}}|\mathcal{F}_t\right]\\ &= \mathbf{E}\left[\int_t^T \lambda e^{-\lambda(s-t)}\max\left\{S_s, y^{\lambda}_s\right\}ds + e^{-\lambda(T-t)}\xi|\mathcal{F}_t\right].\end{aligned}$$

Hence, (2.2) can be further simplified to

$$y^{\lambda}_t = \mathbf{E}\left[\int_t^T e^{-\lambda(s-t)}\left(f_s + \lambda\max\left\{S_s, y^{\lambda}_s\right\}\right)ds + e^{-\lambda(T-t)}\xi|\mathcal{F}_t\right]. \tag{2.5}$$

Given the dynamic programming equation (2.5), we have the following representation result for the value procss $(y^{\lambda}_t)_{0\le t\le T}$, which proves the linear case of Theorem 1.2.

Lemma 2.2 *Suppose that Assumption 1.1 holds, and that $f_s(y,z) = f_s$. Denote $(Y^{\lambda}, Z^{\lambda})$ as the unique solution to the following penalized BSDE:*

$$Y^{\lambda}_t = \xi + \int_t^T f_s ds + \int_t^T \lambda\max\{0, S_s - Y^{\lambda}_s\}ds - \int_t^T Z^{\lambda}_s dW_s.$$

Then $y^{\lambda}_t = Y^{\lambda}_t$ a.s. for $t\in[0,T]$.

Proof. We introduce the dual equation for Y^{λ}:

$$\alpha_t = 1 - \int_0^t \lambda\alpha_s ds \text{ for } t\in[0,T].$$

By applying Itô's formula to $\alpha_t Y^{\lambda}_t$, we obtain

$$\alpha_t Y^{\lambda}_t = \alpha_T Y^{\lambda}_T + \int_t^T \alpha_s\left(f_s + \lambda Y^{\lambda}_s + \lambda\max\{0, S_s - Y^{\lambda}_s\}\right)ds - \int_t^T \alpha_s Z^{\lambda}_s dW_s$$

so that

$$\begin{aligned}Y^{\lambda}_t &= \frac{\alpha_T}{\alpha_t}\xi + \int_t^T \frac{\alpha_s}{\alpha_t}\left(f_s + \lambda\max\left\{S_s, Y^{\lambda}_s\right\}\right)ds - \int_t^T \frac{\alpha_s}{\alpha_t}Z^{\lambda}_s dW_s\\ &= \mathbf{E}\left[e^{-\lambda(T-t)}\xi + \int_t^T e^{-\lambda(s-t)}\left(f_s + \lambda\max\left\{S_s, Y^{\lambda}_s\right\}\right)ds|\mathcal{F}_t\right],\end{aligned}$$

which is exactly dynamic programming equation (2.5). ■

2.2 Representation for Nonlinear Case

In this section, we extend the representation result Lemma 2.2 to the nonlinear case, and give the proof of Theorem 1.2.

Denote (Y^λ, Z^λ) as the unique solution to penalized BSDE (1.3). Consider the optimal stopping time problem (2.1):

$$y_t^\lambda = \mathrm{esssup}_{N\in\mathcal{N}(t,\lambda)}\mathbf{E}\left[\int_t^{T_N\wedge T} f_s(Y_s^\lambda, Z_s^\lambda)ds + S_{T_N}\mathbf{1}_{\{T_N<T\}} + \xi\mathbf{1}_{\{T_N\geq T\}}|\mathcal{F}_t\right].$$

By Lemma 2.2, y^λ admits the following BSDE representation:

$$y_t^\lambda = \xi + \int_t^T f_s(Y_s^\lambda, Z_s^\lambda)ds + \int_t^T \lambda\max\{0, S_s - y_s^\lambda\}ds - \int_t^T z_s^\lambda dW_s.$$

On the other hand, (Y^λ, Z^λ) satisfies penalized BSDE (1.3):

$$Y_t^\lambda = \xi + \int_t^T f_s(Y_s^\lambda, Z_s^\lambda)ds + \int_t^T \lambda\max\{0, S_s - Y_s^\lambda\}ds - \int_t^T Z_s^\lambda dW_s.$$

Define

$$\delta Y_t^\lambda = y_t^\lambda - Y_t^\lambda;\ \delta Z_t^\lambda = z_t^\lambda - Z_t^\lambda.$$

Then $(\delta Y^\lambda, \delta Z^\lambda)$ satisfies the following linear BSDE:

$$\delta Y_t^\lambda = \int_t^T \lambda\beta_s\delta Y_s^\lambda ds - \int_t^T \delta Z_s^\lambda dW_s \tag{2.6}$$

with

$$\beta_s = \frac{\max\{0, S_s - y_s^\lambda\} - \max\{0, S_s - Y_s^\lambda\}}{\delta Y_s^\lambda} \times \mathbf{1}_{\{\delta Y_s^\lambda \neq 0\}}.$$

Obviously, $|\beta_s| \leq 1$, so BSDE (2.6) admits a unique solution (see for example [7] for the proof). On the other hand, $\delta Y_t^\lambda = \delta Z_t^\lambda = 0$ is an obvious solution to BSDE (2.6). Therefore, we conclude that $y_t^\lambda = Y_t^\lambda$ a.s. for $t \in [0, T]$, which proves Theorem 1.2.

3 Optimal Control Representation

Krylov in [12] showed that optimal stopping for controlled diffusion processes can always be transformed to optimal control by using randomized stopping. See also Gyöngy and Siska [8] for its recent development. In this section, our aim is to give optimal control interpretations of both reflected BSDE (1.1) and penalized BSDE (1.3).

Let's first recall the basic idea of Krylov's randomized stopping. For simplicity, we only consider the linear case $f_s(y,z) = f_s$. For any fixed time $t \in [0,T]$, consider a nonnegative control process $(r_s)_{s\geq t}$. Let the payoff functional $\int_t^{\cdot} f_s ds + S_{\cdot}$ stop with intensity $r_s\Delta$ in an infinitesimal interval $(s, s+\Delta)$. Then the probability that stopping does not occur before time s is

$$e^{-\int_t^s r_u du}.$$

The probability that stopping does not occur before time s and does occur in an infinitesimal interval $(s, s+\Delta)$ is

$$e^{-\int_t^s r_u du} r_s \Delta.$$

Therefore, the payoff functional associated with the control process r from $[t,T]$ is given by

$$\int_t^T \left(\int_t^s f_u du + S_s\right) e^{-\int_t^s r_u du} r_s ds + \left(\int_t^T f_u du + \xi\right) e^{-\int_t^T r_u du},$$

where the first term is the payoff if stopping does occur before time T, and the second term corresponds to the payoff if stopping does not occur in the time interval $[t,T]$. By applying integration by parts, the payoff functional is further simplified to

$$\int_t^T (f_s + r_s S_s) e^{-\int_t^s r_u du} + e^{-\int_t^T r_u du} \xi.$$

We have the following optimal control representation for penalized BSDE (1.3).

Proposition 3.1 *Suppose that Assumption 1.1 holds. Denote (Y^λ, Z^λ) as the unique solution to penalized BSDE (1.3). For any fixed time $t \in [0,T]$, define the control set $\mathcal{A}(t,\lambda)$ as*

$$\mathcal{A}(t,\lambda) = \left\{\mathcal{F}_s\text{-adapted process } (r_s)_{s\geq t} : \ r_s = 0 \text{ or } \lambda\right\}.$$

Then the value process of the following optimal control problem:

$$y_t^\lambda = \sup_{r\in\mathcal{A}(t,\lambda)} \mathbf{E}\left[\int_t^T (f_s(Y_s^\lambda, Z_s^\lambda) + r_s S_s) e^{-\int_t^s r_u du} ds + e^{-\int_t^T r_u du} \xi | \mathcal{F}_t\right] \tag{3.1}$$

is given by the solution to penalized BSDE (1.3): $y_t^\lambda = Y_t^\lambda$ a.s. for $t \in [0,T]$. The optimal control is given by $r_s^* = \lambda \mathbf{1}_{\{Y_s^\lambda \leq S_s\}}$.

Proof. We only consider the linear case $f_s(y,z) = f_s$, as the proof for the nonlinear case $f_s(y,z)$ is the same as the proof of Theorem 1.2.

First, analogous to Lemma 2.2, it is easy to show that the expected payoff process associated with any fixed control $r \in \mathcal{A}(t,\lambda)$,

$$y_t^\lambda(r) = \mathbf{E}\left[\int_t^T (f_s + r_s S_s) e^{-\int_t^s r_u du} + e^{-\int_t^T r_u du}\xi | \mathcal{F}_t\right]$$

is the unique solution to the following BSDE:

$$y_t^\lambda(r) = \xi + \int_t^T \left\{ f_s + r_s(S_s - y_s^\lambda(r)) \right\} ds - \int_t^T z_s^\lambda(r) dW_s.$$

Since the control r only appears in the driver, by the BSDE comparison theorem (see for example [7]), $y_t^\lambda = \sup_{r\in\mathcal{A}(t,\lambda)} y_t^\lambda(r)$ satisfies the following BSDE:

$$y_t^\lambda = \xi + \int_t^T \left\{ f_s + \sup_{r_s \in \{0,\lambda\}} r_s(S_s - y_s^\lambda) \right\} ds - \int_t^T z_s^\lambda dW_s.$$

The optimal control is $r_s^* = \lambda \mathbf{1}_{\{y_s^\lambda \le S_s\}}$, and with such an optimal control r^*, the driver becomes $f_s + \lambda \max\{0, S_s - y_s^\lambda\}$, which is the driver of penalized BSDE (1.3). By the uniqueness, $y_t^\lambda = Y_t^\lambda$ a.s. for $t \in [0,T]$. ■

The optimal control representation for reflected BSDE (1.1) is the same as (3.1) except that the control set is change to $\mathcal{A}(t) = \cup_\lambda \mathcal{A}(t,\lambda)$. The value process of the following optimal control problem:

$$y_t = \sup_{r\in\mathcal{A}(t)} \mathbf{E}\left[\int_t^T (f_s(Y_s^\lambda, Z_s^\lambda) + r_s S_s) e^{-\int_t^s r_u du} ds + e^{-\int_t^T r_u du}\xi | \mathcal{F}_t\right] \tag{3.2}$$

is given by the solution to reflected BSDE (1.1): $y_t = Y_t$ a.s. for $t \in [0,T]$.

4 Optimal Switching Representation for Multidimensional Penalized BSDE

Multidimensional Reflected BSDE was firstly introduced by Hamadène and Jeanblanc [9], where they used its solution to characterize the value process of an optimal switching problem, in particular in the setting of power plant management. The related equations were solved by Hu and Tang [11] using penalty method, and by Hamadène and Zhang [10] using iterated optimal stopping time method. See also Chassagneux et al [2] for its recent development. A multidimensional reflected BSDE is a d-dimensional system, where each component $1 \le i \le d$

representing the region i:

$$Y_t^i = \xi^i + \int_t^T f_s^i(Y_s^i, Z_s^i)ds + \int_t^T dK_s^i - \int_t^T Z_s^i dW_s \tag{4.1}$$

under the constraints:

$$\begin{aligned}
\text{(Dominating Condition)}: &\quad Y_t^i \geq \mathcal{M}Y_t^i \text{ for } t \in [0,T], \\
\text{(Skorohod Condition)}: &\quad \int_0^T (Y_t^i - \mathcal{M}Y_t^i)dK_t^i = 0 \text{ for } K^i \text{ continuous and increasing,}
\end{aligned}$$

where

$$\mathcal{M}Y_t^i = \max_{j \neq i}\{Y_t^j - C_t^{ij}\}$$

representing the payoff of switching to region j from region i. The terminal data ξ^i, the driver $f_s^i(y^i, z^i)$ and the switching cost $(C_s^{i,j})_{0 \leq s \leq T}$ are the given data. Different from one-dimensional reflected BSDE whose solution must stay above an obstacle process, the solution of multidimensional reflected BSDE (4.1) evolves in the random closed convex set:

$$\left\{ y \in \mathbb{R}^d : y^i \geq \max_{j \neq i}\{y^j - C_t^{ij}\} \right\}.$$

The following standard assumption on the data set $(\xi^i, f^i, C^{i,j})$ is imposed in Hu and Tang [11]:

Assumption 4.1

- *The terminal data ξ^i and the driver $f_s^i(y^i, z^i)$ satisfy Assumption 1.1;*
- *The switching cost $(C^{ij})_{1 \leq i,j \leq d}$ satisfy (i) $C_t^{ii} = 0$; (ii) $\inf_{t \in [0,T]} C_t^{ij} \geq C > 0$ for $i \neq j$; and (iii) $\inf_{t \in [0,T]} C_t^{ij} + C_t^{jl} - C_t^{il} \geq C > 0$ for $i \neq j \neq l$.*

The condition on the driver $f_s^i(y^i, z^i)$ can be relaxed. For example, in Hamadène and Zhang [10] and Chassagneux et al [2], the driver is even allowed to be coupled in y, i.e. having the form $f_s^i(y, z^i)$.

Under Assumption 4.1, Hu and Tang [11] proved that the solution to multidimensional reflected BSDE (4.1) corresponds to the value process of an optimal switching problem. Indeed, introduce the control set $\mathcal{K}^i(t)$ as

$$\mathcal{K}^i(t) = \left\{ \mathcal{F}_s\text{-adapted process } (u_s)_{s \geq t} : \ u_s = i\mathbf{1}_{\{t\}}(s) + \sum_{k \geq 0} \alpha_k \mathbf{1}_{(\tau_k, \tau_{k+1}]}(s) \right\},$$

where

- $(\tau_k)_{k\geq 0}$ is an increasing sequence of $\mathcal{F}_s$-stopping times valued in $[t,T]$ with $\tau_0 = t$ and $\tau_N = T$ for some integer-valued random variable N.

- $(\alpha_k)_{k\geq 0}$ is a sequence of random variables valued in $\{1,\cdots,d\}$ such that α_k is $\mathcal{F}_{\tau_k}$-measurable, and $\alpha_0 = i$.

Then the value process of the following optimal switching problem:

$$y_t^i = \sup_{u\in\mathcal{K}^i(t)} \mathbf{E}\left[\int_t^T f_s^{u_s}(Y_s^{u_s}, Z_s^{u_s})ds + \xi^{u_T} - \sum_{k\geq 1} C_{\tau_k}^{\alpha_{k-1},\alpha_k}\mathbf{1}_{\{t<\tau_k\leq T\}}|\mathcal{F}_t\right] \tag{4.2}$$

is given by the solution to multidimensional reflected BSDE (4.1): $y_t^i = Y_t^i$ a.s. for $t\in[0,T]$.

On the other hand, Hu and Tang [11] introduced the following multidimensional penalized BSDE to approximate and solve multidimensional reflected BSDE (4.1):

$$Y_t^{i,\lambda} = \xi^i + \int_t^T f_s^i(Y_s^{i,\lambda}, Z_s^{i,\lambda})ds + \int_t^T \lambda\max\{0, \mathcal{M}Y_s^{i,\lambda} - Y_s^{i,\lambda}\}ds - \int_t^T Z_s^{i,\lambda}dW_s, \tag{4.3}$$

and they proved that under Assumption 4.1, $Y^{i,\lambda}$ is increasing in λ, and

$$\lim_{\lambda\uparrow\infty}\mathbf{E}\left[\sup_{t\in[0,T]}|Y_t^{i,\lambda} - Y_t^i|^2 + \int_0^T |Z_t^{i,\lambda} - Z_t^i|^2dt + \sup_{t\in[0,T]}|K_t^{i,\lambda} - K_t^i|^2\right] = 0.$$

Our aim in this section is to give a stochastic control interpretation of multidimensional penalized BSDE (4.3).

Proposition 4.2 *Suppose that Assumption 4.1 holds. Denote $(Y^{i,\lambda}, Z^{i,\lambda})$ as the unique solution to multidimensional penalized BSDE (4.3). For any fixed time $t\in[0,T]$, define the control set $\mathcal{K}^i(t,\lambda)$ as*

$$\mathcal{K}^i(t,\lambda) = \{u\in\mathcal{K}^i(t) : u \text{ is } \mathcal{G}_s\text{-adapted, and } \tau_k = T_k\wedge T \text{ for } k\geq 0\},$$

where $(T_n)_{n\geq 0}$ are the arrival times of a Poisson process $(N_s)_{s\geq t}$ with intensity λ and minimal augmented filtration $\{\mathcal{H}_s\}_{s\geq t}$, and $\mathcal{G}_s = \mathcal{F}_s\vee\mathcal{H}_s$. Then the value process $y^{i,\lambda}$ of the following optimal switching problem:

$$y_t^{i,\lambda} = \sup_{u\in\mathcal{K}^i(t,\lambda)} \mathbf{E}\left[\int_t^T f_s^{u_s}(Y_s^{u_s,\lambda}, Z_s^{u_s,\lambda})ds + \xi^{u_T} - \sum_{k\geq 1} C_{\tau_k}^{\alpha_{k-1},\alpha_k}\mathbf{1}_{\{t<\tau_k\leq T\}}|\mathcal{F}_t\right] \tag{4.4}$$

is given by the solution to (4.3): $y_t^{i,\lambda} = Y_t^{i,\lambda}$ a.s. for $t\in[0,T]$.

Proof. The proof is similar to that of Theorem 1.2, so we only sketch it. The first step is to consider the case that the driver $f_s^i(y^i, z^i)$ is independent of (y^i, z^i), which is then denoted as f_s^i. The key step is to prove the following dynamic programming equation for the value process $y^{i,\lambda}$:

$$y_t^{i,\lambda} = \mathbf{E}\left[\int_t^{\tau_1} f_s^i ds + \max\left\{\mathcal{M}y_{\tau_1}^{i,\lambda}, y_{\tau_1}^{i,\lambda}\right\}\mathbf{1}_{\{\tau_1<T\}} + \xi^i \mathbf{1}_{\{\tau_1=T\}}|\mathcal{F}_t\right], \tag{4.5}$$

which is in the same spirit of dynamic programming equation (2.2) in Lemma 2.1. Indeed, introduce an auxiliary optimal switching problem:

$$\hat{y}_t^{i,\lambda} = \sup_{u\in\hat{\mathcal{K}}^i(t,\lambda)} \mathbf{E}\left[\int_t^T f_s^{u_s} ds + \xi^{u_T} - \sum_{k\geq 1} C_{\tau_k}^{\alpha_{k-1},\alpha_k}\mathbf{1}_{\{t<\tau_k\leq T\}}|\mathcal{F}_t\right], \tag{4.6}$$

where the control set $\hat{\mathcal{K}}^i(t,\lambda)$ is the same as $\mathcal{K}^i(t,\lambda)$ except that we do not require $\alpha_0 = i$. In other words, we are allowed to switch at the starting time t. Due to this additional switching opportunity, we have the following relationship between the two optimal switching problems:

$$\hat{y}_t^{i,\lambda} = \max\left\{\mathcal{M}y_t^{i,\lambda}, y_t^{i,\lambda}\right\}. \tag{4.7}$$

Moreover, if $u_s^* = i\mathbf{1}_{\{t\}}(s) + \sum_{k\geq 0}\alpha_k^*\mathbf{1}_{(\tau_k^*,\tau_{k+1}^*]}(s)$ for $s \geq t$ is the optimal switching strategy for (4.4) starting from time t, by time consistency and (4.7), u_s^* for $s \geq \tau_1$ is also the optimal switching strategy for (4.6) starting from the first possible switching time τ_1.

If u_s^* for $s \geq t$ is the optimal switching strategy for (4.4), by conditional on the first possible switching time τ_1, we have

$$\begin{aligned}
y_t^{i,\lambda} &= \mathbf{E}\left[\mathbf{E}\left[\int_t^T f_s^{u_s^*} ds + \xi^{u_T^*} - \sum_{k\geq 1} C_{\tau_k^*}^{\alpha_{k-1}^*,\alpha_k^*}\mathbf{1}_{\{t<\tau_k^*\leq T\}}|\mathcal{F}_{\tau_1}\right]|\mathcal{F}_t\right] \\
&= \mathbf{E}\Big[\mathbf{E}\Big[\left(\mathbf{1}_{\{t<\tau_1<T,\tau_1\leq\tau_1^*<T\}} + \mathbf{1}_{\{t<\tau_1<T,\tau_1^*=T\}} + \mathbf{1}_{\{\tau_1=T\}}\right) \\
&\qquad \times \left(\int_t^T f_s^{u_s^*} ds + \xi^{u_T^*} - \sum_{k\geq 1} C_{\tau_k^*}^{\alpha_{k-1}^*,\alpha_k^*}\mathbf{1}_{\{t<\tau_k^*\leq T\}}\right)|\mathcal{F}_{\tau_1}\Big]|\mathcal{F}_t\Big] \\
&= \mathbf{E}\Big[\mathbf{1}_{\{t<\tau_1<T\}}\mathbf{E}\left[\mathbf{1}_{\{\tau_1\leq\tau_1^*<T\}}\left(\int_t^{\tau_1} f_s^i ds + \int_{\tau_1}^T f_s^{u_s^*} ds + \xi^{u_T^*} - \sum_{k\geq 1} C_{\tau_k^*}^{\alpha_{k-1}^*,\alpha_k^*}\mathbf{1}_{\{\tau_1<\tau_k^*\leq T\}}\right)|\mathcal{F}_{\tau_1}\right] \\
&\qquad + \mathbf{1}_{\{t<\tau_1<T\}}\mathbf{E}\left[\mathbf{1}_{\{\tau_1^*=T\}}\left(\int_t^{\tau_1} f_s^i ds + \int_{\tau_1}^T f_s^i ds + \xi^i\right)|\mathcal{F}_{\tau_1}\right] \\
&\qquad + \mathbf{1}_{\{\tau_1=T\}}\mathbf{E}\left[\int_t^T f_s^i ds + \xi^i|\mathcal{F}_{\tau_1}\right]|\mathcal{F}_t\Big] \\
&= \mathbf{E}\left[\mathbf{1}_{\{t<\tau_1<T\}}I_1 + \mathbf{1}_{\{t<\tau_1<T\}}I_2 + \mathbf{1}_{\{\tau_1=T\}}I_3|\mathcal{F}_t\right],
\end{aligned}$$

where we used the fact that on $\{\tau_1 = T\}$, we must have $\{\tau_1^* = T\}$. Moreover,

$$I_1 = \mathbf{P}(\tau_1 \leq \tau_1^* < T | \mathcal{F}_{\tau_1}) \int_t^{\tau_1} f_s^i ds + \mathbf{E}\left[\mathbf{1}_{\{\tau_1 \leq \tau_1^* < T\}} \left(\int_{\tau_1}^T f_s^{u_s^*} ds + \xi^{u_T^*} - \sum_{k \geq 1} C_{\tau_k^*}^{\alpha_{k-1}^*, \alpha_k^*} \mathbf{1}_{\{\tau_1 < \tau_k^* \leq T\}}\right) | \mathcal{F}_{\tau_1}\right],$$

and

$$I_2 = \mathbf{P}(\tau_1^* = T | \mathcal{F}_{\tau_1}) \int_t^{\tau_1} f_s^i ds + \mathbf{E}\left[\mathbf{1}_{\{\tau_1^* = T\}} \left(\int_{\tau_1}^T f_s^i ds + \xi^i\right) | \mathcal{F}_{\tau_1}\right].$$

By plugging I_1 and I_2 into the expression for $y_t^{i,\lambda}$, we obtain

$$\begin{aligned}
&\mathbf{E}\Bigg[\mathbf{1}_{\{t<\tau_1<T\}} \int_t^{\tau_1} f_s^i ds \\
&+ \mathbf{1}_{\{t<\tau_1<T\}} \mathbf{E}\left[\mathbf{1}_{\{\tau_1 \leq \tau_1^* < T\}} \left(\int_{\tau_1}^T f_s^{u_s^*} ds + \xi^{u_T^*} - \sum_{k \geq 1} C_{\tau_k^*}^{\alpha_{k-1}^*, \alpha_k^*} \mathbf{1}_{\{\tau_1 < \tau_k^* \leq T\}}\right) + \mathbf{1}_{\{\tau_1^* = T\}} \left(\int_{\tau_1}^T f_s^i ds + \xi^i\right) | \mathcal{F}_{\tau_1}\right] \\
&+ \mathbf{1}_{\{\tau_1 = T\}} \left(\int_t^T f_s^i ds + \xi^i\right) | \mathcal{F}_t\Bigg].
\end{aligned}$$

Note that the second term of the above expression is

$$\sup_{u \in \hat{\mathcal{K}}^i(\tau_1, \lambda)} \mathbf{E}\left[\int_{\tau_1}^T f_s^{u_s} ds + \xi^{u_T} - \sum_{k \geq 1} C_{\tau_k}^{\alpha_{k-1}, \alpha_k} \mathbf{1}_{\{\tau_1 < \tau_k \leq T\}} | \mathcal{F}_{\tau_1}\right],$$

which by definition is $\hat{y}_{\tau_1}^{i,\lambda}$. The result then follows from the relationship (4.7).

The rest of the proof follows the same as the proof for the optimal stopping time representation of one-dimensional reflected BSDE in Theorem 1.2. ■

5 Conclusion

In this paper, we find the stochastic control representations of (multidimensional) reflected BSDE and penalized BSDE, which are summarized in the following table.

Table 1: Stochastic Control Representations of Reflected BSDE and Penalized BSDE

	Optimal stopping representation	Optimal control representation
Reflected BSDE	(1.2) with $\tau \in \mathcal{R}(t)$	(3.2) with $r \in \mathcal{A}(t)$
Penalized BSDE	(1.5) with $\tau \in \mathcal{R}(t, \lambda)$	(3.1) with $r \in \mathcal{A}(t, \lambda)$
Multidimensional Reflected BSDE		(4.2) with $u \in \mathcal{K}^i(t)$
Multidimensional Penalized BSDE		(4.4) with $u \in \mathcal{K}^i(t, \lambda)$

The main feature of the related optimal stopping (switching) representations is that only stopping at arrival times of some exogenous Poisson process is allowed. Such kind of optimal stopping with random intervention times was firstly considered by Dupuis and Wang [4], where they used it to model perpetual American options. Since their problem is one dimensional and with infinite time horizon, they do not even need to introduce penalized equation. Instead, they worked out two ordinary differential equations (*ODE* for short) defined in continuity region and stopping region respectively. Recently, Liang et al [15] find that such kind of optimal stopping with random intervention times can be used to model bank run problems. In the setting of American options, Dai et al [3] intuitively showed that penalty method is closed related to intensity framework. However, they did not introduce any stochastic control interpretations for their penalty method.

Finally, it seems that the only existing representation result for penalized BSDE was given by Lepeltier and Xu [13] and [14][1], where they found a connection between penalized BSDE and a standard optimal stopping problem with modified obstacle $\min\{S_t, Y_t^\lambda\}$. Our represent results are different, and seem more natural: Penalized BSDE is nothing but a random time discretization of an optimal stopping time problem, where the time is discretized by Poisson arrival times.

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